Composite optimization for the resource allocation problem*

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In this paper we consider resource allocation problem stated as a convex minimization problem with linear constraints. To solve this problem, we use subgradient method and gradient descent applied to the dual problem and prove the convergence rate both for the primal iterates and the dual iterates. We also provide economic interpretation for these two methods. This means that iterations of the algorithms naturally correspond to the process of price and production adjustment in order to obtain the desired production volume in the economy. Overall, we show how these actions of the economic agents lead the whole system to the equilibrium.

Keywords: *decentralized pricing; primal-dual method; subgradient method; composite method*

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1 Introduction

In this paper we consider a resource allocation problem in an economy consisting of distributed set of producers which are managed by a centralized price adjustment mechanism. Our approach is based on state-of-the-art convex optimization methods, i.e. we consider the resource allocation as an optimization problem, solve it by a variant of Nesterov's accelerated gradient method, provide convergence analysis, and give an economic interpretation of the steps of this method.

The problem of optimal resource allocation is to maximize producers' aggregated profits by sharing available resources. Popularized and advocated mainly in the monograph [1], the mechanisms of decentralized resource allocation gained a lot of attention in economics and operations research since then, see e.g. [2, 3, 6] and references therein. Each producer seeks to maximize its own profit and, in total, all the producers need to produce a certain amount of products. This problem can be cast as an optimization problem with the objective corresponding to the cost function of all producers and constraints corresponding to the condition for the necessary volume of production. We assume that constraints are linear and separable. In this optimization problem, primal variables are production bundles and dual variables represent prices of resources.

In this paper, following [9], we consider a different price adjustment mechanism. In particular, we consider the resource allocation problem without centralized price control, each factory setting up its own price. Each factory produces one product and sets the price for selling the product to the Center. The Center knows the amount which needs to be produced by all the factories in total, selects the most advantageous offers (i.e. selects the offers with the best price)

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and tries to purchase the product in the required volume. Factories adjust the volume of product and the prices, based on the volume bought from them by the Center and the demand from the Center for this particular factory. Under an additional assumption of strongly convexity of the primal objective, we consider the dual optimization problem as a composite minimization problem, meaning that the objective in the dual problem is a sum of two functions a smooth and a simple non-smooth. We use gradient descent to obtain faster convergence rates for the resource allocation problem, namely, we obtain rate $O(1/t)$.

The paper is organized as follows. In Section 1 we consider the primal problem and describe its economical interpretation. In Section 2 we describe the method of subgradient projection for the resource allocation problem and give the interpretation for the step. In Section 3 we use composite gradient method for resource allocation problem and obtain estimation for the convergence rate. In Section 4 we consider accelerated gradient descent.

2 Problem statement

In this paper we consider the following resource allocation problem. Suppose that there is a Center and n factories which produce one product. Each factory has its own cost function $f_k(x_k)$, $k = 1, \ldots, n$ representing the total cost of production of the volume $x_k \in \mathbb{R}$, the number of tons of product produced by the factory k in one year. Since each factory has its own owner, the cost functions of the factories are unknown to the Center, and each factory knows only its own function. Each factory is also entitled to set its own price for product. In this statement, the price does not affect the quality of the product in any way, i.e. all factory produce the same product, only at different prices. The Center buys product from the factories. Also the Center needs that all factories produce not less than C tons of this product in total per year. So, the Center determines y_k - the amount of tons of product that will be purchased from the factory k. Since each factory wants to reduce costs, we can write down the following resource allocation problem

$$(P) \qquad f(\mathbf{x}) = \sum_{k=1}^{n} f_k(x_k) \to \min_{\substack{\sum\limits_{k=1}^{n} y_k \geqslant C,\ x_k \geqslant y_k;\\ y_k \geqslant 0,\ x_k \geqslant 0,\ k=1,\ldots,n}},$$

where cost functions $f_k(x_k)$ $k = 1, \ldots, n$ are increasing and μ-strongly convex functions, i.e. $f_k''(x) \geqslant 0\, \forall x \geqslant 0,\ k = 1, \ldots, n$.

Remark 1. Assumption of μ-strong convexity of functions $f_k(x_k)$, $k = 1, \ldots, n$ holds, for example, when these functions are twice continuously differentiable and have positive second derivative. Economically this means that the production cost grows faster than the volume of the production. In other words, the production cost of a new unit volume grows as the volume of production grows. For example, this happens for Agriculture. If the producer grows wheat, then the more he wants to produce from one hectare, the more he should invest in fertilizers, chemicals from pests, or even genetic technology. For a factory the producer has to invest more and more in more advanced facilities such as robots, production machines, etc.

Introducing dual variables p_k, $k = 1, ..., n$ and using the duality theory, we obtain

$$\min_{\substack{\sum_{k=1}^{n} y_k \geqslant C,\, x_k \geqslant y_k,\, y_k \geqslant 0; \\ x_k \geqslant 0,\, k=1,\,\dots,\,n}} f(\mathbf{x}) = \min_{\substack{\sum_{k=1}^{n} y_k \geqslant C\; y_k \geqslant 0; \\ x_k \geqslant 0,\, k=1,\,\dots,\,n}} \left\{ f(\mathbf{x}) + \sum_{k=1}^{n} \max_{p_k \geqslant 0} p_k (y_k - x_k) \right\}$$
$$= - \min_{p_1,\,\dots,\,p_n \geqslant 0} \left\{ \sum_{k=1}^{n} \max_{x_k \geqslant 0} (p_k x_k - f_k(x_k)) - \min_{\sum_{k=1}^{n} y_k \geqslant C;\, y_k \geqslant 0} \sum_{k=1}^{n} p_k y_k \right\}$$
$$= - \min_{p_1,\,\dots,\,p_n \geqslant 0} \left\{ \sum_{k=1}^{n} \max_{x_k \geqslant 0} (p_k x_k - f_k(x_k)) - C \min_{k=1,\,\dots,\,n} p_k \right\}$$
$$= - \min_{p_1,\,\dots,\,p_n \geqslant 0} \left\{ \sum_{k=1}^{n} \left\{ p_k x_k(p_k) - f_k(x_k(p_k)) \right\} - C \min_{k=1,\,\dots,\,n} p_k \right\},$$

where

$$x_k(p_k) = \arg\max_{x_k \geqslant 0} \left\{ p_k x_k - f_k(x_k) \right\}, \quad k = 1,\, 2,\, \dots,\, n. \tag{1}$$

Then the dual problem (up to a sign) has the following form

$$(D) \qquad \varphi(p_1,\, \dots,\, p_n) = \sum_{k=1}^{n} \left\{ p_k x_k(p_k) - f_k(x_k(p_k)) \right\} - C \min_{k=1,\,\dots,\,n} p_k \to \min_{p_1,\,\dots,\,p_n \geqslant 0.}$$

Note that, the Slater constraint qualification condition holds for the primal problem (P). Thus, the strong duality holds and both the primal problem (P) and the dual problem (D) have solutions.

3 Subgradient descent

For the sake of completeness, in this section, we consider dual problem (D) as a non-smooth optimization problem and apply subgradient method to solve it with the rate $O(1/\sqrt{t})$. We also provide an economic interpretation of the numerical procedure based on subgradient method. The material of this section is not new and mostly follows [5]. Nevertheless, it is convenient to consider it here. Later we will compare the convergence rate and interpretation with a faster approach based on gradient descent.

The subgradient of the objective function in the dual problem D can be written in the following form

$$\nabla \varphi(p_1,\, \dots,\, p_n) = \begin{pmatrix} x_1(p_1) \\ \cdot \\ \cdot \\ \cdot \\ x_n(p_n) \end{pmatrix} - C \begin{pmatrix} \lambda_1 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_n \end{pmatrix}, \tag{2}$$

where $\sum\limits_{k=1}^{n} \lambda_k = 1,\ \lambda_k \geqslant 0$ if $k \in \arg\min\limits_{j=1,\,\dots,\,n} p_j$ and $\lambda_k = 0$, if $k \notin \arg\min\limits_{j=1,\,\dots,\,n} p_j$.

To solve this problem we use the method of subgradient projection. Where in our case, since we have a condition $\mathbf{p} \geqslant \mathbf{0}$, the projection is just the positive cut. So, we have the following step of the method

$$\mathbf{p}^{t+1} = \left[\mathbf{p}^t - h \nabla \varphi(\mathbf{p}^t) \right]_+, \tag{3}$$

Input: h – stepsize, $\mathbf{p}^0$ – starting point.

1. Set p_k^t, $k = 1, \ldots, n$ and calculate
$$x_k(p_k^t) = \arg\max_{x_k \geqslant 0} \left\{ p_k^t x_k - f_k(x_k) \right\}, \quad k = 1, 2, \ldots, n.$$
2. Form the vector $\lambda(\mathbf{p}^t)$ as $\lambda^t = (\lambda_1^t, \ldots, \lambda_n^t)^\top$, where $\sum\limits_{k=1}^{n} \lambda_k^t = 1$, $\lambda_k^t \geqslant 0$, and if $\lambda_k^t > 0$, when $k \in \arg\min\limits_{j=1,\ldots,n} p_j^t$.
3. Do the step
$$\mathbf{p}^{t+1} = \left(\mathbf{p}^t - h\nabla\varphi(p_1^t, \ldots, p_n^t)\right)_+,$$
where
$$\nabla\varphi(p_1^t, \ldots, p_n^t) = \mathbf{x}(\mathbf{p}^t) - C\lambda(\mathbf{p}^t).$$

Figure 1 General projected subgradient method

where h is the step, which we determine later. Note that here and in subsequent arguments we mean that $\nabla\varphi(\mathbf{p})$ is a subgradient (vector), arbitrarily chosen from the convex compact set — subdifferential. In this case, which subgradient from a particular subdifferential will be chosen is not significant, since this will not affect the estimate of the convergence rate.

3.1 Algorithm of subgradient descent for the resource allocation problem

In this subsection we describe the algorithm for the behavior of the Center and factories.

Under $C\lambda_k(\mathbf{p}^t)$ is understood the volume of the Center's purchase from the k-th factory on t iteration, since that $\lambda_k(\mathbf{p}^t)$ determines which share of the total amount C Center is going to purchase from the k-th factory on t iteration.

Each k-th component of the subgradient (2) can be interpreted as the difference between the production $x_k(p_k)$ of the k-th factory from the volume $C\lambda_k$ of the Center's demand for this factory. Taking this into account, we propose the following interpretation for determining the vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^\top$. After the Center has received information about prices and the optimal volumes of production from factories, it finds the minimum among them. Then the Center allocates the desired total amount of purchase between factories which have the price equal to the minimum one. For some factories, for which $C\lambda_k$ is positive, it can happen that $x_k(p_k) - C\lambda_k < 0$. This is a signal for the k-th producer that the demand exceeds the supply and the k-th price can be increased together with the increase of the produced volume.

Using this, we can rewrite algorithm in our case as fig. 2.

Those each factory counts how much its production differs from the desired volume of the Center's purchase from this factory this year. And if the Center does not buy anything from the factory or buys less than it produced, then the factory lowers the price. If the Center is ready to buy more than the factory produced, the factory raises the price. In the case of equality, the factory does not change anything.

Input: $\varepsilon > 0$ – accuracy, $\mathbf{p}^0$ – starting point.

1. Set the stepsize $h = \frac{\varepsilon}{nC^2}$.
2. Given the price vector $\mathbf{p}^t$ for the current year, producers calculate the optimal production plan for these prices as
$$x_k(p_k^t) = \arg\max_{x_k \geqslant 0} \left\{ p_k^t x_k - f_k(x_k) \right\}, \quad k = 1, 2, \ldots, n.$$
and communicates this information to the Center.
3. The Center determines the shares of purchases for each producer, i.e. forms a vector $\lambda(\mathbf{p}^t)$ as $\lambda^t = (\lambda_1^t, \ldots, \lambda_n^t)^\top$, where $\sum\limits_{k=1}^{n} \lambda_k^t = 1$, $\lambda_k^t \geqslant 0$ if $k \in \arg\min\limits_{j=1,\ldots,n} p_j$ and $\lambda_k^t = 0$, if $k \notin \arg\min\limits_{j=1,\ldots,n} p_j^t$ and sends this vector to all factories.
4. Each factory adjusts the price for the next year as follows
$$\mathbf{p}^{t+1} = \left(\mathbf{p}^t - h(\mathbf{x}(\mathbf{p}^t) - C\lambda(\mathbf{p}^t)) \right)_+ .$$

Figure 2 Subgradient method for the resource allocation

3.2 Slater's condition

In this section we estimate the upper bound for the solution of the dual problem (D). To do this, we use the Slater's condition and carry out arguments analogous to [7]. Set $\bar{x}_1 = \ldots = \bar{x}_n = \frac{2C}{n}$ and $\bar{y}_1 = \ldots = \bar{y}_n = \frac{C}{n}$. And define $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_n)^\top$ and $\bar{\mathbf{y}} = (\bar{y}_1, \ldots, \bar{y}_n)^\top$. Notice that the point $\left(\bar{\mathbf{x}}^\top, \bar{\mathbf{y}}^\top\right)^\top$ satisfies the Slater's condition as $\frac{2C}{n} = \bar{x}_k > \bar{y}_k = \frac{C}{n}$, $k = 1, \ldots, n$.

Because the cost functions $f_k(x_k)$, $k = 1, \ldots, n$ are increasing due to the economic interpretation, we obtain

Then, we can formulate the following theorem about convergence rate

Theorem 1 ([5]). Let Algorithm 1 be run with starting point p^0 satisfying $0 \leqslant p_k^0 \leqslant p_{max}$, $k = 1, \ldots, n$ for
$$N = \left\lceil \frac{164(Cnp_{max})^2}{\varepsilon^2} \right\rceil$$
steps. Then
$$f(\mathbf{x}^N) - f(\mathbf{x}^*) \leqslant \varepsilon,\; C - \sum_{k=1}^{n} x_k^N \leqslant \frac{\varepsilon}{3p_{max}}, \tag{5}$$
where $\mathbf{x}^N = \frac{1}{N} \sum\limits_{t=0}^{N-1} \mathbf{x}(\mathbf{p}^t)$

Note that the number of iterations N to achieve accuracy ε is very large. To improve the number of iterations, in the following sections we consider the methods based on the composite optimization approach.

$$\begin{aligned}
\sum_{k=1}^{n} f_k(0) &= \min_{\substack{\sum_{k=1}^{n} y_k \geqslant C,\, y_k \geqslant 0,\\ x_k \geqslant 0,\, k=1,\dots,n}} \sum_{k=1}^{n} f_k(x_k) \\
&= \min_{\substack{\sum_{k=1}^{n} y_k \geqslant C,\, y_k \geqslant 0,\\ x_k \geqslant 0,\, k=1,\dots,n}} \left\{ \sum_{k=1}^{n} f_k(x_k) + \sum_{k=1}^{n} (y_k - x_k) \underbrace{p_k}_{=0} \right\} \\
&\leqslant \max_{p \geqslant 0} \min_{\substack{\sum_{k=1}^{n} y_k \geqslant C,\, y_k \geqslant 0,\\ x_k \geqslant 0,\, k=1,\dots,n}} \left\{ \sum_{k=1}^{n} f_k(x_k) + \sum_{k=1}^{n} (y_k - x_k)\, p_k \right\} \\
&= \min_{\substack{\sum_{k=1}^{n} y_k \geqslant C,\, y_k \geqslant 0,\\ x_k \geqslant 0,\, k=1,\dots,n}} \left\{ \sum_{k=1}^{n} f_k(x_k) + \sum_{k=1}^{n} (y_k - x_k)\, p_k^* \right\} \\
&\leqslant \sum_{k=1}^{n} f_k(\bar{x}_k) + \sum_{k=1}^{n} (\bar{y}_k - \bar{x}_k)\, p_k^* \\
&\leqslant \sum_{k=1}^{n} f_k(\bar{x}_k) - \frac{C}{n} \sum_{k=1}^{n} p_k^*.
\end{aligned}$$

Since $p_k^* \geqslant 0,\ k = 1, \ldots, n$, we can formulate the following Lemma:

Lemma 1. Let the $\mathbf{p}^*$ be a solution to the dual problem (D). Then it satisfies the inequality

$$\|\mathbf{p}^*\|_1 \leqslant p_{\max},$$

where

$$p_{\max} = \frac{n}{C} \left(\sum_{k=1}^{n} f_k\left(\frac{2C}{n}\right) - \sum_{k=1}^{n} f_k(0) \right).$$

Due to the Lemma 1 we obtain that

$$p_k^* \leqslant p_{\max},\ k = 1, \ldots, n,$$

from which we obtain the following equation

$$\|\mathbf{p}^*\|_2 \leqslant \sqrt{n} p_{\max}. \tag{4}$$

4 Composite gradient method for the resource allocation problem

In this section we consider the non-accelerated composite method to solve the dual problem D. We describe this method and give an economical interpretation for the step of the method. Also we estimate the convergence rate of the proposed algorithm.

So, lets consider this problem as the composite optimization problem. We can rewrite the dual problem as

$$\varphi(p_1, \ldots, p_n) = \psi(p_1, \ldots, p_n) + g(p_1, \ldots, p_n),$$

where

$$\psi(p_1, \ldots, p_n) = \sum_{k=1}^{n} \Big\{ p_k x_k(p_k) - f_k(x_k(p_k)) \Big\} = \langle \mathbf{p}, \mathbf{x}(\mathbf{p}) \rangle - f(\mathbf{x}(\mathbf{p})) \tag{6}$$

is convex function, which gradient satisfies the Lipschitz condition for all variables $\mathbf{p}^1, \mathbf{p}^2 \geqslant 0$

$$\left\| \nabla\psi(\mathbf{p}^1) - \nabla\psi(\mathbf{p}^2) \right\|_2 \leqslant L_\psi \left\| \mathbf{p}^1 - \mathbf{p}^2 \right\|_2. \tag{7}$$

And the convex non smooth composite function

$$g(p_1, \ldots, p_n) = -C \min_{k=1, \ldots, n} p_k.$$

To solve the problem we use the method with the following step

$$\mathbf{p}^{t+1} = \arg\min_{\mathbf{p}\geqslant 0} \left\{ \left\langle \nabla\psi(\mathbf{p}^t), \mathbf{p} - \mathbf{p}^t \right\rangle - C \min_{k=1, \ldots, n} p_k + \frac{L_\psi}{2} \left\| \mathbf{p} - \mathbf{p}^t \right\|_2^2 \right\}. \tag{8}$$

To obtain the solution of the (8) we can formulate the following Lemma.

Lemma 2. We can determine solution of the step (8) as follows

– If $\sum\limits_{k=1}^{n} \left(-\tilde{p}_k^{t+1}\right)_+ > \dfrac{C}{L_\psi}$ then $p_{\text{center}}^{t+1} = 0$ and

$$p_k^{t+1} = \max\left(0,\, \tilde{p}_k^{t+1}\right), \ k = 1, \ldots, n.$$

– Else $p_{\text{center}}^{t+1} > 0$ and determines from

$$\sum_{k=1}^{n} \left(p_{\text{center}}^{t+1} - \tilde{p}_k^{t+1}\right)_+ = \frac{C}{L_\psi}$$

and

$$p_k^{t+1} = \max\left(p_{\text{center}}^{t+1},\, \tilde{p}_k^{t+1}\right), \ k = 1, \ldots, n,$$

where $\tilde{\mathbf{p}}^{t+1} = \mathbf{p}^t - \dfrac{1}{L_\psi}\mathbf{x}(\mathbf{p}^t)$.

4.1 Algorithm of composite gradient method for the resource allocation problem

In this subsection we give an economical interpretation for the step (8). And describe the algorithm for the behavior of the Center and factories.

Using the Lemma 2 we can economically interpret step in following case. Each year t factory k predict the price $\tilde{p}_k^{t+1} = p_k^t - \dfrac{1}{L_\psi} x_k(p_k^t)$ for product for the next year. In this way factory k, in accordance with the price and volume for the current year, determines the lowest price for which it is willing to produce and sell product in $t+1$ year. So, sell at a lower price for this factory is not profitable. After this, all the factories send vector of prices $\tilde{\mathbf{p}}^{t+1}$ to the Center. Then, the Center analyzes this information and determines the price p^{t+1}_{center} which we can interpret as the price at which the Center will purchase product for the next year. So next year $t+1$ the Center planning to purchase wheat from factories which set the price not more than p^{t+1}_{center}. Then the Center send this price to all factories. After that, all the factories which were going to produce sell product at a price lower than the Center's price, raise the price to p^{t+1}_{center}, since even at such price the Center will buy from them.

To obtain more economical interpretation of the step, lets rewrite the step (8) as

$$\mathbf{p}^{t+1} = \left[\mathbf{p}^t - \frac{1}{L_\psi}\left(\mathbf{x}(\mathbf{p}^t) - C\lambda(\mathbf{p}^{t+1})\right)\right]_+, \tag{9}$$

where $\lambda^{t+1} = \lambda(\mathbf{p}^{t+1})$ and $\sum\limits_{k=1}^{n} \lambda^t_k = 1$, $\lambda^t_k \geqslant 0$, and if $\lambda^t_k > 0$, then $k \in \operatorname*{arg\,min}\limits_{j=1,\ldots,n} p^t_j$. We can compare this step with the step (3) and notice that they have the same structure but the step (9) of the considered algorithm is a function given in an implicit form. Similarly to section 1 we can interpret the step as follows. After each factory k set the price p^t_k for the current year t and determined the optimal production plan $x_k(p^t_k)$ for the setting price, each factory report this information to the Center. Having received this information, the Center should inform each factory the volume of purchase from this factory for the next year $C\lambda(\mathbf{p}^{t+1})$. But since the quality of product is equal for each company the Center wants to buy product only from factories producing wheat at the lowest price. So in fact the Center needs to predict the prices for the next year and proceeding from this determine from which factories and how much it will purchase. Formally, on each iteration the Center needs to find a fixed point of a multivalued function (9).

Since we find the solution of the (8), this is also a solution of the (9). And since the step (8) we can rewrite as

$$\mathbf{p}^{t+1} = \left[\tilde{\mathbf{p}}^{t+1} + \frac{C}{L_\psi}\lambda(\mathbf{p}^{t+1}))\right]_+,$$

the components of the vector $\lambda(\mathbf{p}^{t+1})$ satisfy

$$\lambda_k(\mathbf{p}^{t+1}) = \frac{L_\psi}{C}\left(p^{t+1}_{\text{center}} - \tilde{p}^{t+1}_k\right)_+.$$

So, using this, we obtain the Algorithm **??**.

4.2 Estimation of the convergence rate

In this subsection we consider the convergence rate of the proposed algorithm. To formulate the main result about convergence rate we need the following.

Put the vector $\mathbf{p}^0 = (p^0_1, \ldots, p^0_n)^\top$, where $\mathbf{p}^0$ — vector of initial prices. And its components such as

$$0 \leqslant p^0_k \leqslant p_{\max},\ k = 1, \ldots, n,$$

then we obtain that

$$\left\|\mathbf{p}^0\right\|_2 \leqslant \sqrt{n}p_{\max}.$$

Let us introduce a set

$$B^+_{3R}(0) = \{\mathbf{p} \,:\, \mathbf{p} \geqslant 0,\ \left\|\mathbf{p} - \mathbf{p}^0\right\|_2 \leqslant 3R\},$$

Input: $N > 0$ – number of steps, L_ψ – Lipschitz constant of gradient ψ, $\mathbf{p}^0$ – starting point.

1. Knowing the prices p_k^t, $k = 1, \ldots, n$ for the current year t, producers calculate the optimal plan for the production according these prices
$$x_k(p_k^t) = \arg\max_{x_k \geqslant 0} \left\{ p_k^t x_k - f_k(x_k) \right\}, \quad k = 1, 2, \ldots, n.$$
2. The Center forms a prediction for the lowest possible producers' prices vector for the next year
$$\tilde{p}_k^{t+1} = p_k^t - \frac{1}{L_\psi} x_k(p_k^t), \quad k = 1, 2, \ldots, n.$$
3. The Center determines the price p_{center}^{t+1} at which it will purchase product for the next year $t+1$ and sends this price to all factories.
   - If $\sum\limits_{k=1}^n \left(-\tilde{p}_k^{t+1}\right)_+ > \dfrac{C}{L_\psi}$ then $p_{center}^{t+1} = 0$;
   - Else $p_{center}^{t+1} > 0$ and solves equaltion
$$\sum_{k=1}^n \left(p_{center}^{t+1} - \tilde{p}_k^{t+1}\right)_+ = \frac{C}{L_\psi}$$
4. Each producer adjusts the price for the next year as follows
$$p_k^{t+1} = \max\left(p_{center}^{t+1}, \tilde{p}_k^{t+1}\right), \; k = 1, \ldots, n.$$

Figure 3 Composite gradient method for the resource allocation

where
$$\left\|\mathbf{p}^0 - \mathbf{p}^*\right\|_2 + \left\|\mathbf{p}^0\right\|_2 \leqslant 2\left\|\mathbf{p}^0\right\|_2 + \|\mathbf{p}^*\|_2 = 3p_{\max}\sqrt{n} = R, \tag{10}$$
herewith all the obtaining $\mathbf{p}^t$ will consist in $B_{2R}^+(\mathbf{0})$:
$$\|\mathbf{p}^t\|_2 \leqslant 2R, \tag{11}$$
since (second paragraph [4])
$$\|\mathbf{p}^t\|_2 = \|\mathbf{p}^t - \mathbf{p}^0\|_2 + \left\|\mathbf{p}^0\right\|_2 \leqslant \|\mathbf{p}^t - \mathbf{p}^*\|_2 + \|\mathbf{p}^* - \mathbf{p}^0\|_2 + \left\|\mathbf{p}^0\right\|_2$$
$$\leqslant 2\|\mathbf{p}^* - \mathbf{p}^0\|_2 + \left\|\mathbf{p}^0\right\|_2 = 2\|\mathbf{p}^*\|_2 + 3\left\|\mathbf{p}^0\right\|_2 = 5p_{\max}\sqrt{n} \leqslant 2R.$$

Define $\mathbf{p}^N = \frac{1}{N}\sum\limits_{t=1}^N \mathbf{p}^t$ and $\mathbf{x}^N = \frac{1}{N}\sum\limits_{t=0}^{N-1} \mathbf{x}(\mathbf{p}^t)$. Since that, the main result can be formulated as the following theorem:

Theorem 2. Let it be necessary to solve problem (P) in the following sense

$$f(\mathbf{x}^N) - f(\mathbf{x}^*) \leqslant \varepsilon,\ C - \sum_{k=1}^{n} x_k^N \leqslant \frac{\varepsilon}{9p_{\max}}. \tag{12}$$

To solve this problem, we consider the dual problem (D), which we solve by the composite optimization. As a stopping criterion we choose condition for the duality gap and the discrepancy in the constraint

$$f(\mathbf{x}^N) + \varphi(\mathbf{p}^N) \leqslant \varepsilon,\ C - \sum_{k=1}^{n} x_k^N \leqslant \frac{\varepsilon}{9p_{\max}},$$

from which by the weak duality we obtain the necessary conditions (12). Then Algorithm will stop not later than after N iterations which is determined as

$$N = \frac{82 L_\psi R^2}{9\varepsilon}.$$

5 Concluding Remarks

In this paper we considered resource allocation problem stated as a convex minimization problem with linear constraints. We use subgradient method and gradient descent applied to the dual problem and prove the convergence rate both for the primal iterates and the dual iterates. We compared convergence rate for this two methods. We also provide economic interpretation for these two methods. And proof our result by numerical experiments.

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Композитная оптимизация для задачи распределения ресурсов*

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В этой статье мы рассматриваем проблему выделения ресурсов, сформулированную как выпуклая задача минимизации с линейными ограничениями. Чтобы решить эту проблему, мы используем субградиентный метод и градиентный спуск, примененный к двойственной задаче. Мы также предоставляем экономическую интерпретацию для этих двух методов. Это означает, что итерации алгоритмов естественно соответствуют процессу корректировки цены и производства, чтобы получить желаемый объем производства в экономике. В целом, мы показываем, как эти действия экономических агентов приводят всю систему к равновесию.

Ключевые слова: *децентрализованное ценообразование; прямо-двойственный метод; субградиентный метод; композиционный метод*

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